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THE VIBRATION OF A ROTATING HEAVY NON-UNIFORM THREAD AND ITS STABILITY†

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The linear vibration of a heavy non-uniform thread is investigated for different boundary conditions at the ends and taking an arbitrary additional tension into account. The thread is assumed to be ideal and inextensible and the motion takes place in a plane which may rotate about the vertical axis at constant angular velocity. A general scheme for solving the initial- and boundary-value problem is proposed. Attention is focused mainly on the effective computation of the natural frequencies and mode of vibration. Given specific parametric types of mass distribution for the thread, sufficiently complete families of solutions describing the principal modes of vibration are constructed. Based on these families, stability and instability domains are constructed effectively, in terms of the system parameters, for a plane vibration of a rotating heavy thread subject to concentrated tension. New mechanical effects, of possible interest in practice, are observed and discussed.

1. FORMULATION OF THE PROBLEM

Consider the small plane transverse vibration of a rotating heavy non-uniform thread subject to an arbitrary additional tension. The conditions governing the attachment of the thread above and below are represented in the general form of non-homogeneous boundary conditions of the third kind (elastic attachment). As a result we arrive at the following boundary-value problem

$$\rho(x)\ddot{u} = (W(x)u')' + \rho(x)\omega^2 u + f(x,t), \quad 0 < x < l, \quad u = u(x,t) \tag{1.1}$$

$$W(x)u'(x,t) \mp k_x u(x,t) = \mp h_x(t), \quad x = 0, l \tag{1.2}$$

where u denotes the transverse displacements of the thread elements in a plane rotating about the vertical x axis, dots denote derivatives with respect to time t, primes denote derivatives with respect to the Euler coordinate x, l is the length of the thread, $\rho(x)$ denotes the linear density, $W(x)$ is the total tension in the cross-section corresponding to the x coordinate, ω is the angular velocity of rotation of the plane (ω = const), $f(x, t)$ is the distributed external force, $k_{0,l}$ are the coefficients of elasticity of the attachment and $h_{0,l}$ are the external force effects, concentrated at $x = 0, l$. The function $\rho(x)$ is assumed to be continuous and separable from zero: $0 < \rho_1 \leqslant \rho(x) \leqslant \rho_2 < \infty$; the functions $f(x, t)$ and $h_{0,l}(x)$ are also assumed to be sufficiently smooth in their domains of definition.

It will be assumed below that the total tension $W(x)$ of the thread in (1.1) and (1.2) is due to two factors: the weight of a segment of the thread, $m(x)g$, and an additional load W_0 concentrated at the lower end, so that the total tension may be expressed as

$$W(x) = m(x)g + W_0, \quad m(x) = \int_0^x \rho(s)ds, \quad W_0 \geqslant 0, \quad g > 0 \tag{1.3}$$

Here g is the acceleration due to gravity (if necessary, the variability of this acceleration may be taken into consideration) and W_0 is a force concentrated at the lower or (and) upper end ($x = 0$) (the weight of a load acting through a block, or a force of some other physical nature—elastic, electromagnetic, etc.). The extension of the thread will be neglected, since normally $W(x) \ll ES(x)$, where E is Young's modulus of the material and $S(x)$ is the area of cross-section. An investigation of the combined transverse and longitudinal vibrations require a separate study. In order to determine the motion of system (1.1)–(1.3), the initial distributions of the displacements and velocities of the points of the thread must be given

$$u(x,0) = u^0(x), \quad \dot{u}(x,0) = \dot{u}^0(x), \quad 0 \leqslant x \leqslant l \tag{1.4}$$

The functions u^0 and $\dot{u}^0$ in (1.4) are assumed to be sufficiently smooth for a strong (physical) solution

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of the initial- and boundary-value problem (1.1)–(1.4) to exist. It is required to construct such a solution and to investigate its properties, depending on the parameters of the system.

Studies of the vibration of a heavy thread taking into account additional tensions (both longitudinal and transverse) are of considerable interest, both theoretically and in practice; see [1–5] and other literature. It is particularly important to analyse the natural vibration, based on solving the boundary-value problem of finding the eigenvalues and eigenfunctions of an equation with variable coefficients [1–9].

Remark. The mathematical model of the vibration of a heavy thread in an arbitrary plane, when the thread is not rotating (the classical case), is obtained from the above by setting $\omega = 0$ in Eq. (1.1). The small vibration of a free rotating thread (i.e. a thread not subject to plane constraints) in the general case may be three-dimensional due to Coriolis forces of inertia. In a rotating system, these are described by the following vector relations (see (1.1), (1.2) and (1.4))

$$\rho(x)\ddot{\mathbf{u}} = (W(x)\mathbf{u}')' + \rho(x)(\omega^2\mathbf{u} - 2[\boldsymbol{\omega} \times \dot{\mathbf{u}}]) + \mathbf{f}(x,t)$$

$$W(x)\mathbf{u}'(x,t) \mp k_x\mathbf{u}(x,t) = \mp\mathbf{h}_x(t), \quad x = 0, l \tag{1.5}$$

Using an orthogonal transformation, that is, changing to a non-rotating reference system, we obtain a boundary-value problem equivalent to the case $\omega = 0$. Simultaneously, the vectors $\mathbf{f}$, $\mathbf{h}_{0,l}$, $\mathbf{u}_0$, $\dot{\mathbf{u}}^0$ must be appropriately transformed. Indeed, we have

$$\rho(x)\ddot{\mathbf{U}} = (W(x)\mathbf{U}')' + \mathbf{F}(x,t,\varphi), \quad \mathbf{u} = \Pi(\varphi)\mathbf{U}$$

$$[W(x)\mathbf{U}' \mp k_x\mathbf{U}]_{x=0,l} = \mp\mathbf{H}_{0,l}, \quad \mathbf{H}_{0,l} = \Pi^{-1}(\varphi)\mathbf{h}_{0,l}(t) \tag{1.6}$$

$$\mathbf{F} = \Pi^{-1}(\varphi)\mathbf{f}, \quad \Pi^{-1}(\varphi) = \Pi(-\varphi) = \Pi^{\mathrm{T}}(\varphi), \quad \varphi = \omega t$$

Here $\Pi(\varphi)$ is the rotation matrix. As a result we obtain a system of two boundary-value problems (1.6). Each problem, for the components U_1 and U_2 of the vector $\mathbf{U}$, can be solved independently, since the components of the vectors $\mathbf{F}$, $\mathbf{H}_{0,l}$, $\mathbf{U}^0$, $\dot{\mathbf{U}}^0$ are known. The homogeneous boundary-value problems and the corresponding Sturm–Liouville problems do not contain the parameter ω in the coefficients of U and $\dot{U}$, that is, they are equivalent to the case $\omega = 0$ in (1.1). System (1.1), (1.2) may therefore be considered to be more meaningful and general than the vector analogue (1.5), (1.6) just considered. Note that these problems are essentially different in their formulations and the solutions may be qualitatively different.

A boundary-value problem analogous to (1.6) may be obtained by introducing the complex variable $x = u_1 - iu_2$ and substituting $z = w\exp(i\omega)$. The unknown complex function $w(x, t)$ is described by relations of the same form as (1.1), (1.2) (setting $\omega = 0$ in the coefficient of u in the vibration equation (1.1)). Of course, the transformed functions f and $h_{0,l}$ will contain the factor $\exp(-i\varphi)$.

2. THE SOLUTION OF THE INITIAL- AND BOUNDARY-VALUE PROBLEM

We will apply the standard approach (the Fourier method), based on constructing a solution of the homogeneous boundary-value problem (1.1), (1.2) (with $f = h_{0,l} \equiv 0$), since it admits of separation of variables. To determine the eigenvalues (natural frequencies) and eigenfunctions (natural modes), we obtain a modified self-adjoint boundary-value problem (the Sturm–Liouville problem [6–9])

$$(W(x)X')' + \Lambda\rho(x)X = 0, \quad \Lambda = \lambda + \omega^2$$

$$W(x)X'(x) \mp k_xX(x) = 0, \quad x = 0, l \tag{2.1}$$

Here Λ and $X(x)$ are an unknown eigenvalue and eigenfunction of problem (2.1) and λ is the separation parameter. It is required to construct a complete denumerable family of orthonormal functions $\{X_n(x)\}$ (basis) and a set of numbers $\{\Lambda_n\}$ [6–9]. The solution of this problem is of fundamental interest from an applied point of view; see below.

Let us suppose that such families are known; we then obtain a denumerable linear system of independent equations and initial conditions for the Fourier coefficients Θ_n ($n = 1, 2, \ldots$)

$$\ddot{\Theta}_n + \lambda_n\Theta_n = f_n(t) + X_n(0)h_0(t) + X_n(l)h_l(t), \quad f_n = (f, X_n)$$

$$\Theta_n(0) = (u^0, \; X_n)_\rho, \quad \dot{\Theta}(0) = (\dot{u}^0, X_n)_\rho, \quad u(x,t) = \sum_{n=1}^{\infty}\Theta_n(t)X_n(x) \tag{2.2}$$

where the parentheses denote scalar products.

A solution $\Theta_n(t)$ of the Cauchy problem (2.2) can be constructed by elementary means, using simple quadratures. It will be assumed in what follows that the free motions of the thread (that is, with $f = h_{0,l} \equiv 0$) are vibratory, that is, the smallest eigenvalue λ_1 is positive. The situation in which $\lambda_1 \leqslant 0$ may be of interest for applications, since then the plane vibration of the rotating thread becomes unstable. Such a state of the motion means that the displacements and velocities of the thread elements increase at an exponential or linear rate. In that case, non-linear factors, dissipation, etc. must be taken into account.

From the theoretical and applied points of view, it is most important to carry out a high-accuracy investigation of the behaviour of the lower frequencies of free vibrations, in particular, of the fundamental frequency [6–9]. The aim of the subsequent study is to achieve a sufficiently accurate computation and detailed analysis of the first eigenvalue λ_1 as a function of the parameters of system (1.1), (1.2). To that end we will use the efficient numerical-analytical method of accelerated convergence, based on the variational approach to the solution of the Sturm–Liouville problem (2.1) and the differential relation established between the eigenvalue λ_1 and the parameter l—the length of the thread [8].

3. THE VARIATIONAL APPROACH TO THE SOLUTION OF THE STURM-LIOUVILLE PROBLEM

Let us express Eqs (2.1) in the form of a variational isoperimetric problem [6–9]

$$\Phi[X] = \int_0^l W(x)X'^2 dx - W(x)X'X\,|_0^l \to \min_X$$

$$I[X] = \|X\|^2 = (X,X)_\rho = 1, \quad [W(x)X'(x) \mp k_x X(x)]_{x=0,l} = 0 \tag{3.1}$$

The absolute minimum of the functional $\Theta[X] = \Lambda$ in problem (3.1) is the first eigenvalue Λ_1, and the differentiable function $X(x, \Lambda)$ at which this minimum is reached is the first eigenfunction $X_1(x) = X(x, \Lambda_1)$. If $W(x) \geqslant W_0 > 0$, a solution of the variational problem exists and is unique, and then $\Lambda_1 > 0$ for all $k_{0,\,l} \geqslant 0$. Only in the limiting case $k_{0,l} = 0$ (both ends of the thread are free) do we obtain $\Lambda_0 = 0(\lambda_0 = -\omega^2 \leqslant 0)$, while $X_0(x) \equiv 1/\sqrt{(m(l))}$, where $m(l)$ is the mass of the thread.

The subsequent eigenvalues Λ_n and eigenfunctions $X_n(t)$, $n \geqslant 2$, are defined by relations (3.1) and the additional conditions that the desired functions be orthogonal with weight $\rho(x)$ to the previous functions, as expressed by the following recurrence relation

$$I_k[X] = (X, X_{k-1}(x))_\rho = 0, \quad k = 1,2,\ldots n \tag{3.2}$$

The functions $X_1, X_2, \ldots, X_{n-1}$ in (3.2) are assumed to be known, having been constructed at the previous steps. The general properties of the numbers Λ_n as a function of the index n and of the functions $X_n(x)$ as functions of n and x have been investigated in detail in [6–9].

The Sturm–Liouville problem (2.1) is the boundary-value problem corresponding to the Euler–Lagrange equation. It determines the necessary and sufficient conditions for the functional Φ of the variational problem (3.1), (3.2) to have a minimum; Λ is a Lagrange multiplier. Note that the terminal term in (3.1) is also positive-definite for $k_{0,l} > 0$.

Using (3.1), one can obtain an upper bound Λ_1^* for the first eigenvalue Λ_1 (the Rayleigh principle [6–9])

$$0 < \Lambda_1 \leqslant \Lambda_1^* = \Phi[\psi]/I[\psi], \quad [W(x)\psi'(x) \mp k_x \psi(x)]_{x=0,l} = 0 \tag{3.3}$$

where $\psi(x)$ is a continuously differentiable "trial" function (or comparison function) satisfying boundary conditions (3.3) and chosen from general physical considerations regarding the first mode of vibrations (often in the form of trigonometric or polynomial functions); see below. Computational practice has shown [6–9] that even a very rough selection of the function $\psi(x)$ in (3.3) yields satisfactory results: the initial estimate Λ_1^* differs from the exact value with a relative error ranging from 1% to 10%. Given such an accuracy, one can use a procedure to increase the accuracy of Λ_1, obtaining a very precise result (a relative error of 10^{-4}–10^{-8}) at a quite moderate computational cost; see Sections 4–6 below. As observed in Section 2, the rapidly converging method of accelerated (quadratic) convergence was developed to that end. If the initial approximation to Λ_1 (or to $\Lambda_2, \Lambda_3, \ldots$) is not precise enough, one can use other methods (the Rayleigh–Ritz method, the finite-element method continuation with respect to a parameter, etc.). For a rigorous justification and convergence rate estimate for the Rayleigh–Ritz method one can consult Krylov [6] and others. Higher-order eigenvalues Λ_n ($n \gg 1$) may be calculated using high-precision asymptotic formulae [9], which may be used even for relatively small n ($n \geqslant 5$) and in some cases yield satisfactory results even for $n \geqslant 2$.

4. THE METHOD OF ACCELERATED CONVERGENCE

We will present the basic principle of the method (algorithm) as applied to the Sturm–Liouville problem (2.1) using the variational relations (3.1)–(3.3) (see [8]). To fix our ideas, we will consider the case $n = 1$, which is of special interest for applications, in particular, for investigating the stability of the plane vibration of a rotating heavy thread taking into account an additional tension. Henceforth we will simplify the notation by omitting the subscript.

Thus, suppose that some method, such as the Rayleigh–Ritz method or continuation with respect to a parameter (see Sections 5 and 6), has been used to obtain an approximation Λ_0 (in particular, $\Lambda^0 = \Lambda^*$ (3.3)). We can then formulate a Cauchy problem for Eq. (2.1) [8]

$$(W(x)v')' + \Lambda^0\rho(x)v = 0, \quad v(0) = \alpha_0 l, \quad v'(0) = \beta_0, \quad \alpha_0 + \beta_0 = 1$$
$$\alpha_0 = d_0(d_0 + k_0)^{-1}, \quad \beta_0 = k_0(d_0 + k_0)^{-1}, \quad d_0 = W_0 l^{-1} \tag{4.1}$$

We will assume from now on that a solution $\upsilon(x, \Lambda^0)$, $\upsilon'(x, \Lambda^0)$ of this Cauchy problem is known with sufficient accuracy in numerical or analytical form, or as a computational procedure. It will automatically satisfy the zero boundary condition (2.1) at $x = 0$. The dependence of the functions υ and υ' on the other parameters (k_o, d_0, W_0, g and those occurring in $\rho(x)$) will not be indicated at the moment, in order to simplify the notation; see below.

We now calculate the function $E(x, \Lambda^0)$, requiring it to vanish at some minimum value of $x = \xi^0 > 0$, that is, we find the first positive root, ξ^0 of the equation

$$E(x, \Lambda^0) = \alpha_l l\nu(x)v'(x, \Lambda^0) + \beta_l v(x, \Lambda^0) = 0, \quad \alpha_l + \beta_l = 1 \tag{4.2}$$
$$\alpha_l = d_l(d_l + k_l)^{-1}, \quad \beta_l = k_l(d_l + k_l)^{-1}, \quad d_l = W(l)l^{-1}$$
$$\nu(x) \equiv W(x)W(l)^{-1}, \quad \xi^0 = \xi(\Lambda^0) = \min \arg_x E(x, \Lambda^0) > 0$$

The relation for ξ^0 in (4.2) is constructed during the numerical integration of the Cauchy problem (4.1). If Λ^0 is sufficiently close to Λ, the root ξ^0 exists and is simple. As a measure of this closeness, δ, we take a numerical parameter ε

$$\varepsilon = (l - \xi^0)l^{-1}, \quad 0 \leqslant |\varepsilon| \ll 1, \quad \delta = (\Lambda - \Lambda^0)\Lambda^{-1}, \quad 0 \leqslant |\delta| \ll 1 \tag{4.3}$$

We have $\varepsilon = 0$ if and only if $\delta = 0$. In what follows, we will assume that, given Λ^0, the value of the root ξ^0 and the parameter ε of (4.3) have been found with sufficient accuracy. In the procedure presented here, the solution of problem (4.1)–(4.3) turns out to be one of the main and more costly operations (in computational terms).

We will use perturbation methods [10, 11] to approximate the solution of the initial Sturm–Liouville problem (2.1). According to previous work [8, 11], we obtain an improved value $\Lambda^{(1)}$ in the first approximation with respect to ε (with error $O(\varepsilon^2)$)

$$\Lambda^{(1)} = \Lambda^0 + \varepsilon\mu(\xi^0, \Lambda^0), \quad \xi^0 = \xi(\Lambda^0), \quad \mu < 0 \tag{4.4}$$
$$\mu(\xi^0, \Lambda^0) = -\xi^0 W(\xi^0)v'^2(\xi^0, \Lambda^0)\|v_0\|^{-2} - \Lambda^0\xi^0\rho(\xi^0)v^2(\xi^0, \Lambda^0)\|v_0\|^{-2}$$

where $\| \upsilon_0 \|$ is the mean-square norm with weight $\rho(x)$ of the function $\upsilon_0 = \upsilon(x, \Lambda^0)$, defined by analogy with (2.2) in the interval $0 \leqslant x \leqslant \xi^{(0)}$. By (4.4), the quantity $\mu(\lambda, \Lambda)l^{-1}$ is the derivative of the eigenvalue Λ with respect to l; obviously, it is strictly negative. This means that increasing the length l of the thread (string) decreases the frequency of natural vibration $(\Lambda - \omega^2)^{1/2}$ and decreasing it increases the frequency.

This process, computing progressively better approximations to Λ, may be continued indefinitely. We now use the improved value $\Lambda^{(1)}$ as the initial approximation, as Λ^0 was used, in relations of the same type as (4.1)–(4.4); we again obtain an improved value $\Lambda^{(2)} = \Lambda^{(1)} + \varepsilon^{(1)}\mu(\xi^{(1)}, \Lambda^{(1)})$, where $\varepsilon^{(1)}$

corresponds to the root $\xi^{(1)} = \xi(\Lambda^{(1)})$; and so on. As a result, we obtain the following recurrence procedure for improving the accuracy of the eigenvalue Λ and the eigenfunction X, which has the property of accelerated (quadratic) convergence with respect to the original small parameter ε. The algorithm is described by the following relations

$$\Lambda^{(k+1)} = \Lambda^{(k)} + \varepsilon^{(k)}\mu(\xi^{(k)}, \lambda^{(k)}), \ \varepsilon^{(k)} = (l - \xi^{(k)})l^{-1}, \ k = 0,1,2,\ldots$$
$$\xi^{(k)} = \xi(\Lambda^{(k)}) = \min \arg_x E(x, \Lambda^{(k)}) > 0, \ \xi^{(0)} = \xi^0 = \xi(\Lambda^0), \ \varepsilon^{(0)} = \varepsilon \qquad (4.5)$$
$$E(x, \Lambda^{(k)}) \equiv \alpha_l l \upsilon(x)\upsilon'(x, \Lambda^{(k)}) + \beta_l \upsilon(x, \Lambda^{(k)}) = 0$$
$$(W(x)\upsilon')' + \Lambda^{(k)}\rho(x)\upsilon = 0, \ \upsilon(0) = \alpha_0 l, \ \upsilon'(0) = \beta_0, \ \upsilon_{(k)} = \upsilon(x, \Lambda^{(k)})$$

The functions $\upsilon_{(k)}$, $\upsilon'_{(k)}$, E are obtained by numerical integration of the Cauchy problem (4.5), that is, of problem (4.1) for $\Lambda^0 = \Lambda^{(k)}$; it is assumed that $\Lambda^{(k)}$ was evaluated in the previous iteration. The procedure (4.5) yields the following error estimates at the $(k + 1)$th step [8]

$$|\Lambda^{(k+1)} - \Lambda| \leq C_\Lambda \varepsilon^{(k+1)}, \ |\varepsilon^{(k)}| \leq d(c\varepsilon)^{\theta(k)}, \ \theta(k) = 2^k, \ k = 0,1,2,\ldots$$
$$\max_x(|X(x,\Lambda) - \upsilon(x, \Lambda^{(k)})| + l|X'(x,\Lambda) - \upsilon'(x, \Lambda^{(k)})| \leq C_X \varepsilon^{(k)} \qquad (4.6)$$
$$0 \leq x \leq \max(l, \xi^{(k)}), \ C_{\Lambda,X}, c, d = \text{const}$$

The constants $C_{\Lambda,X}$, c and d may be effectively estimated based on the properties of the function $\rho(x)$ and the quantities l, W_0, g and $k_{0,l}$. Thus, algorithm (4.5) yields accelerated convergence (quadratic with respect to ε) of the unknown quantities according to (4.6). Computational practice has shown the process to be extremely efficient: it is simple to implement, stable to noise and does not lead to rounding-off error accumulation. Two to four iterations are usually sufficient to determine practically precise values of $\Lambda, X(x), \|X\|, X'$ for fairly rough choices of the trial functions $\psi(x)$ (3.3). The efficiency of the computations increases markedly if the convergence acceleration algorithm is combined with the procedure of continuation with respect to the system parameters, depending on whose values the required solution Λ, $X(x)$ is constructed (see below). Note that the accuracy of computations using (4.5) must be compatible with the iteration index k; there is no point in carrying out excessively accurate computations.

The subsequent numbers Λ_n and functions $X_n(x)$, $n \geq 2$ are determined according to the scheme outlined above. The difference is that the root of the equation $E(x, \Lambda_n^{(k)}) = 0$ (see (4.2) and (4.5)) to be calculated with the requisite accuracy is the nth root $\xi_n(\Lambda_n^{(k)})$ rather than the first root, as in the case considered above.

To evaluate the quadratures for the mean-square norm with weight $\rho(x)$, $\|\upsilon_{(k)}\|$, in Eqs (4.4) and (4.5), one usually uses some very accurate method to integrate functions defined by numerical solution of Cauchy problems, such as Simpson's method. Instead of this process, it is possible to compute the functions $w = \partial\upsilon/\partial\Lambda$, $w' = dw/dx$ by combined integration of the Cauchy problem (4.1) for υ and υ' and the following Cauchy problem for w and w' and the following Cauchy problem for w and w'

$$(W(x)w')' + \Lambda^{(k)}\rho(x)w = -\rho(x)\upsilon, \ w(0) = w'(0) = 0 \qquad (4.7)$$
$$\|\upsilon_k\|^2 \equiv \int_0^{\xi^{(k)}} \upsilon^2(x, \Lambda^{(k)})\rho(x)dx =$$
$$= W(\xi^{(k)})[w(\xi^{(k)}, \Lambda^{(k)})\upsilon'(\xi^{(k)}, \Lambda^{(k)}) - \upsilon(\xi^{(k)}, \Lambda^{(k)})w'(\xi^{(k)}, \Lambda^{(k)})]$$

In computations, instead of integrating the system of equations (4.1) and (4.7), it is more convenient to use systems of equations of standard Cauchy form

$$\upsilon' = y/W(x), \ y' = -\Lambda^{(k)}\rho(x)\upsilon, \ \upsilon(0) = \alpha_0 l, \ y(0) = \beta_0 W_0$$
$$w' = z/W(x), \ z' = -\Lambda^{(k)}\rho(x)w - \rho(x)\upsilon, \ w(0) = z(0) = 0 \qquad (4.8)$$

suitably transforming the expressions for $E(x, \Lambda^{(k)})$ and $\|\upsilon_{(k)}\|^2$.

In conclusion, it should be noted that in all the expressions (4.4)–(4.7) one can set $\xi^{(k)} = l$ without loss of accuracy in powers of $\varepsilon^{(k)}$, except the expressions for determining the quantity $\varepsilon^{(k)}$ (see (4.3), (4.5)).

We will now consider specific examples of linear densities $\rho(x)$ and boundary conditions, that is, values of the parameters $\alpha_{0,l}$, $\beta_{0,l}$.

5. THE NATURAL VIBRATION OF A UNIFORM HEAVY ROTATING THREAD SUBJECT TO TENSION

If it is assumed that the linear density function is constant, ρ = const, the Cauchy problem (4.5) becomes

$$(\sigma(x,\chi)v')' + \gamma v = 0,\ \gamma = \gamma^{(k)} = \chi\Lambda^{(k)}\Omega^{-2},\ v(0) = \alpha_0,\ v'(0) = \beta_0 \tag{5.1}$$

$$\sigma(x,\chi) = 1 + \chi(x-1),\ \chi = mg(W_0 + mg)^{-1},\ 0 < \chi \le 1,\ \Omega^2 = gl^{-1}$$

The function υ and the argument x ($0 \leqslant x \leqslant 1$) are normalized by the value of l, $m = \rho l$. According to (5.1), the equation determining the abscissa $\xi^{(k)}$ (4.5) normalized by l may be written as

$$E(x,\gamma^{(k)},\chi) = \alpha_l\sigma(x,\chi)v'(x,\gamma^{(k)},\chi) + \beta_l v(x,\gamma^{(k)},\chi) = 0 \tag{5.2}$$

$$\xi^{(k)} = \xi(\gamma^{(k)},\chi),\ \alpha_l,\ \beta_l \ge 0,\ \alpha_l + \beta_l = 1$$

It is convenient to use Eqs (5.1) and (5.2) in procedure (4.5) together with the method of continuation with respect to the parameter χ, that is, to look for $\gamma = \gamma(\chi)$, $\upsilon = \upsilon(x, \chi)$. In the limit of negligibly small weight mg of the thread relative to W_0 (the mass occurs only in γ), we obtain the model of a string with elastically attached ends. The Cauchy problem is integrable by elementary means in trigonometric functions. To determine $\gamma(0)$, we have to solve a transcendental equation for $\nu = \sqrt{(\gamma(0))}$

$$\mathrm{tg}\nu = \nu(\alpha_0\beta_l + \beta_0\alpha_l)(\nu^2\alpha_0\alpha_l - \beta_0\beta_l)^{-1},\ \gamma_n = \nu_n^2,\ n \ge 1 \tag{5.3}$$

The value of ν_n depends on the two parameters α and β, where $\alpha = (\alpha_0, \alpha_l)$, $\beta = (\beta_0, \beta_l)$ are related to one another by the normalization conditions $\alpha + \beta = (1, 1)$. In particular, when $\alpha_0 = 0, 1$ ($\beta_0 = 1, 0$) and $\alpha_l = 0, 1$ ($\beta_l = 1, 0$), we obtain the well-known elementary cases of attached or free ends. In the general situation, the required roots $\nu_n^* = \nu_n(\alpha, \beta)$ of Eq. (5.3) may be found numerically. The eigenvalues $\gamma_n(0)$ and eigenfunctions $X_n(x, 0)$ have the following form ($n = 1, 2, \ldots$)

$$\gamma_n(0) = \nu_n^2,\ X_n(x) = \alpha_0\cos\nu_n x + \beta_0\nu_n^{-1}\sin\nu_n x \tag{5.4}$$

As observed previously, the subject of most interest for applications is that of the lower frequencies and modes of vibration, in particular, $n = 1$.

The value of $\gamma(0)$ as in (5.4), which was calculated from (5.3), will be used below as an initial estimate γ^0 when $\chi = \chi_1 > 0$, where χ_1 is sufficiently small, as may be established by numerical experiment. Integrating the Cauchy problem (5.1) for $\gamma = \gamma^0$ and determining the necessary root $\xi^{(0)} = \xi(\gamma^0, \chi_1)$ from (5.2), we then use formulae (4.3), (4.4) and (4.7) to find an improved value of $\xi^{(1)}(\chi_1)$, and then, applying the recurrence procedure, we obtain a high-accuracy approximation $\gamma(\chi_1)$. We then implement the process of continuation with respect to the parameter χ: $0 < \chi_1 < \chi_2 < \ldots < \chi_i < 1$, where i is sufficiently large.

It is well known that Eq. (5.1) can be reduced to a Bessel equation of order zero [1, 3, 12]: replacing the argument x in one-to-one fashion by θ, we can use a standard procedure to obtain the required Bessel equation and its general solution υ_0

$$\begin{gathered} v'' + \theta^{-1}v' + \eta^2 v = 0,\ v = v(\eta\theta),\ \eta^2 = 4\gamma\chi^{-1},\ 0 < \chi < 1 \\ v_0 = AJ_0(z) + BN_0(z),\ z = \eta\theta,\ A, B = \text{const} \\ \theta = \sigma^{1/2}(x,\chi),\ \theta_1 \le \theta \le 1,\ \theta_1 = (1-\chi)^{1/2},\ 0 < \theta_1 < 1 \end{gathered} \tag{5.5}$$

where the prime denotes derivatives with respect to θ. Detailed analytical, numerical and graphical data for the Bessel functions $J_\nu(z)$ and Neumann functions $N_\nu(z)$ of order zero ($\nu = 0$) and other orders may be found in [12]. Using the boundary conditions for $\theta = \theta_1$, $\theta = 1$ (see (5.1) and (5.2)), we obtain a characteristic equation for the quantity $\eta = \eta(\theta_1, \alpha, \beta)$, which is related to $\gamma = \gamma(\chi, \alpha, \beta)$ by (5.5). This equation also contains derivatives of the functions J_0, N_0 : $J_0' = -J_1$, $N_0' = -N_1$ [12].

Note that $\theta_1 \to 1$ as $\chi \to 0$ (the model of a string), that is, the substitution $\chi \to \theta$ (5.5) becomes degenerate. This situation was considered above (see (5.3) and (5.4)). For small χ $(0 < \chi \ll 1)$, one can deal with problem (5.1), (5.2) by the method of regular perturbations, for which a generating solution is known. The approximate analytical investigation presents no special difficulty or interest. We will now consider the general case $0 \leqslant \chi \leqslant 1$; letting $\chi \to 1$ $(\theta_1 \to 0)$ we obtain regular singular points $x = 0$, $\theta = 0$ in Eqs (5.1) and (5.5), respectively. This situation requires an asymptotic analysis, which is more conveniently performed on the basis of Eq. (5.5) and the functions J_0 and N_0.

Investigations of solutions of the problem for arbitrary admissible $\alpha_{0,l}$, $\beta_{0,l}$ are extremely cumbersome. We will therefore consider the special cases of boundary conditions corresponding to the limiting values of $\alpha_{0,l}$, $\beta_{0,l}$, paying particular attention to an analysis of the first mode of vibration.

1. Let $\alpha_l = \beta_0 = 0$ $(\alpha_0 = \beta_l = 1)$; this is the classical case of the vibration of a heavy non-uniform thread (chain) with an additional tension. Figure 1 (curve 1) shows the function $\gamma(\chi)$, the mechanical interpretation of which (5.1) is more intuitive than that of $\eta(\theta_1)$. It follows from the graph that $\gamma(0) = \pi^2/4$, and $\gamma(\chi)$ decreases monotonically as $\chi \to 1$; the minimum is $\gamma(1) = \eta_0^2/4$, where $\eta_0 = \eta(0) \approx 2.4048$ is the first root of the function $J_0(\eta)$ [1, 12]. Asymptotic analysis of the root $\eta(\theta)$ for $0 \leqslant \theta_1 \ll 1$ yields an approximate expression $\eta(\theta_1) \approx \eta_0 + O(\theta_1^2)$, from which it follows that $\gamma(\chi) = \gamma(1) + O((1-\chi))$, that is, the function γ tends linearly to $\gamma(1)$ when $0 \leqslant 1-\chi \ll 1$. In general, too, $\gamma(\chi)$ decreases in a practically linear fashion (see Fig. 1).

The family of normalized eigenfunctions $V(x, \chi) = \upsilon \| \upsilon \|^{-1}$ is represented in Fig. 2 by the dashed curves; curves 1–3 correspond to the values $\chi = 0.9999$, $\chi = 0.9$ and $\chi = 0$. At $\chi = 0$ (curve 3) one has the vibration of a tautly stretched string with free lower end; the situation $\chi \approx 1$ (curve 1) corresponds to the vibration of a heavy thread without additional tension [1], whose first mode is described by the function $J_0(\eta_0\theta) = J_0(2\sqrt{\gamma(1)}x)$. Curve 2, for the intermediate value $\chi = 0.9$, is shown in order to clarify the evolution of the family as χ varies from $\chi = 0$ to $\chi = 1$.

2. An analogous analysis may be carried out for the "inverse" case $\alpha_0 = \beta_l = 0$ $(\alpha_l = \beta_0 = 1)$, that is, the upper end is free and the lower end is fixed. A graph of the function $\gamma(\chi)$ is shown in Fig. 1 (curve 2). Note that for small χ curves 1 and 2 are close to each other, but as $\chi \to 1$ they separate and one observes a qualitative difference: $\gamma \to 0$ as $\chi \to 1$. The decrease turns out to be very slow; when $\chi = 0.9999$ one has $\gamma \sim 0.1$. Asymptotic analysis yields the estimate $\eta(\theta_1) = O((\ln\theta_1^{-1} - 1/2)^{-1/2})$, $\theta_1 \ll 1$, which, for the desired value of γ is reduced to the form $\gamma(\chi) \approx O((\ln(1-\chi)^{-1} - 1)^{-1})$, where $0 < 1 - \chi \ll 1$.

Figure 2 shows the normalized eigenfunctions $V(x, \chi)$ (the dot–dash curves); curves 4–6 correspond to the χ values specified in (1) above. Curve 6 is analogous to the curve for case 3 of the vibration of a string with free end. As $\chi \to 1$ (curve 4), the function $V(x, \chi)$ tends weakly to a discontinuous function of the Heaviside type, indicating that a thread with free upper end and without additional tension cannot perform a natural vibration. The functions $\gamma(\chi)$ and $V(x, \chi)$, where $0 < \chi \ll 1$, can easily be investigated by a perturbation method.

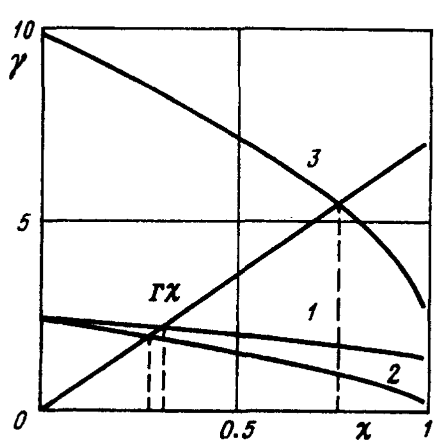

Fig. 1.

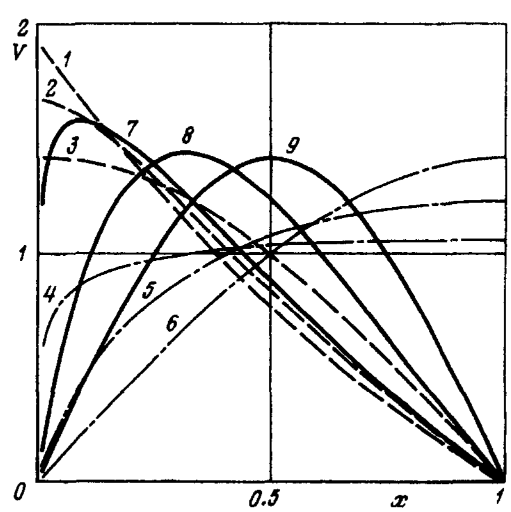

Fig. 2.

3. The vibration of a heavy thread with additional tension when both ends are fixed, that is, $\alpha_{0,l} = 0$ ($\beta_{0,l} = 1$), is of particular interest. A graph of the function $\gamma(\chi)$ is shown in Fig. 1 (curve 3). This function decreases quite rapidly from $\gamma(0) = \pi^2$ to $\gamma(1) = \eta_0^2/4 \approx 1, 4$, with the rate of decrease increasing (in absolute value) without limit as $\chi \uparrow 1$. Comparison with case 1 shows that curve 3 remains above curve 1 when $0 \leqslant \chi < 1$. Thus, at non-zero tension, the frequency of the natural vibration of a "string" exceeds that of a thread with a free end. Asymptotic analysis using Bessel and Neumann functions yields the approximate expression $\eta(\theta_1) = \eta_0 + O(\ln^{-1}(1-\chi)^{-1})$, from which it also follows that curve 3 will be above curve 1 for all $0 \leqslant \chi < 1$.

The normalized eigenfunctions $V(x, \chi)$ are shown in Fig. 2 (the solid curves). When $\chi = 0$ (curve 9) one obtains the classical case of the first mode of a uniform tautly stretched string. As $\chi \uparrow 1$ (curve 7), the eigenfunction tends (weakly) to a discontinuous function, which is identical in form when $x > 0$ with $J_0(2\sqrt{(\gamma(1)x)})$ (apart from a normalizing factor; see case 1). It is clear, however, that the "string" cannot vibrate without additional tension.

4. The case of a heavy stretched thread free for transverse displacements ($\beta_{0,1} = 0, \alpha_{0,1} = 1$) obviously leads to zero values of $\eta(\theta_1) = \gamma(\chi) \equiv 0$ for all admissible values $0 \leqslant \theta_1, \chi \leqslant 1$ and to a mode $V(x, \chi) \equiv 1$ (see (5.1) and (5.5)). The next root of the characteristic equation will be non-zero. For this root we have $\eta(0) = 3.8317$ ($J_1(\eta) = 0$).

Let us investigate the stability of the vibration of a thread for the boundary conditions of cases 1–4 on the basis of the graphs of the functions $\gamma(\chi)$ (Fig. 1). These values and the families of normalized functions $V(x, \chi)$ were constructed by the accelerated convergence method of Section 4, using the scheme (5.1), (5.2) and the procedure of continuation with respect to the parameter χ. It is extremely difficult to use Bessel and Neumann functions to find the desired solution by numerical means. Computational practice shows that a rapidly converging method for constructing the eigenvalues and eigenfunctions is more efficient, even in the case considered of a "known exact solution" in special functions (see below, Section 6).

A sufficient condition for the plane vibration of a rotating heavy thread with additional tension to be stable is the inequality $\lambda > 0$. It follows from the definition of the parameter γ (5.1) that the following inequality holds (the stability condition for a vibration with respect to the particular norm chosen)

$$\gamma(\chi) > \Gamma\chi,\ 0 \leq \chi \leq 1,\ \Gamma = \omega^2\Omega^{-2},\ \gamma = (\lambda + \omega^2)\chi\Omega^{-2} \tag{5.6}$$

Fix some value of $\Gamma \geqslant 0$. Then the thread will perform a stable motion at those values of χ for which the points on the curves $\gamma(\chi)$ lie above the straight line Γ_χ. In cases 1–3 such values of χ, $0 \leqslant \chi < \chi^* \leqslant 1$, exist. A geometric interpretation of condition (5.6) is presented in Fig. 1. In addition, in cases 1 and 3, when $0 \leqslant \Gamma < \gamma(1) \approx 1.4$, this inequality holds for all χ, $0 \leqslant \chi \leqslant 1$. As Γ increases, the length of the interval $0 \leqslant \chi \leqslant \chi^*$ decreases, in such a way that $\chi \to 0$ as $\Gamma \to \infty$, which is natural. In case 4 ($\gamma(\chi) \equiv 0$) the "vibration" is always unstable: when $\Gamma = 0$ ($\omega = 0$) the instability is secular; when $\Gamma > 0$ ($\omega \neq 0$) it is exponential. These conclusions are in agreement with mechanical considerations.

6. THE VIBRATION OF A NON-UNIFORM THREAD

Using the accelerated convergence method of Section 4, we will investigate the free vibration of a heavy non-uniform string whose total mass is fixed and equal to m_*. As observed, to fix our ideas, we will confine our attention to the high-accuracy computation of the first eigenvalue and eigenfunction, considering two examples.

6.1. *A linearly varying mass per unit length*. Suppose the linear density ρ, mass m and tension W have the form

$$\begin{aligned} &\rho(x) = \rho_0(1 - \varkappa(2x - l)l^{-1}) \\ &m(x) = \rho_0 x(1 - \varkappa(x - l)l^{-1}),\ -1 < \varkappa < 1,\ m(l) = \rho_0 l = m_* \\ &W(x) = W_0 + P_0 - P_0[1 - xl^{-1}(1 + \varkappa - \varkappa x l^{-1})],\ P_0 = m_* g \end{aligned} \tag{6.1}$$

It then follows from (6.1) that $m(l) = m_*$ and that the mass is independent of the parameter κ characterizing the coefficient of variation of the linear density in the admissible range: $\rho(x) > 0$ for all $0 \leqslant x \leqslant l$; at $x = l/2$ or $\kappa = 0$ the quantity $\rho \equiv \rho_0$ is independent of the values of κ or x. Using formulae (6.1), we transform to a dimensionless argument x and dimensionless parameters γ, κ and χ in the Sturm–Liouville problem (2.1), obtaining the following equation

$$(p(x)X')' + \gamma r(x)X = 0,\ \gamma = \chi\Lambda\Omega^{-2},\ \Omega^2 = g/l$$
$$0 \le x \le 1,\ p(x) = 1 - \chi(1 - x(1 + \varkappa - \varkappa x)),\ -1 < \varkappa < 1 \tag{6.2}$$
$$r(x) = 1 - 2\varkappa(x - 1/2),\ \chi = P_0(W_0 + P_0)^{-1},\ 0 \le \chi < 1$$

This equation contains the two given parameters κ and χ and the unknown parameter γ, which is to be determined.

The parameter χ in (6.2) characterizes the ratio of the weight of the thread P_0 to the total tension at the point $x = 1$. Note that $|\kappa| < 1$; as $\kappa \to \pm 1$ the inextensibility condition may be violated. The boundary conditions take the form

$$\alpha_x p(x)X'(x) \mp \beta_x X(x) = 0,\ \alpha_x \ge 0,\ \beta_x \ge 0,\ \alpha_x + \beta_x = 1, x = 0,1 \tag{6.3}$$

We will now consider conditions of the first kind at one or both ends, by analogy with cases 1–3 in Section 5. In this case, the eigenvalues γ of problem (6.2), (6.3) will depend on two parameters $\gamma = \gamma(\kappa, \chi)$. The problem considered in Section 5 corresponds to the case $\kappa = 0$. We then apply the accelerated convergence method (Section 4) and the procedure of continuation with respect to the parameters κ and χ. As initial exact value of γ, used also as the estimate γ^0, we take $\gamma^0 = \gamma(0, 0) = \pi^2/4$ (cases 1 and 2) and $\gamma^0 = \pi^2$ (case 3). Case 4 (both ends free) will not be considered, for the reason outlined in Section 5 (instability of the motion). Modern computers enable us to make a highly efficient computation of the surface $\gamma = \gamma(\kappa, \chi)$ using the accelerated convergence algorithm, as well as a graphical construction of a suitable projection of the surface. However, to determine the numerical data it proves more convenient to determine the surface as a sufficiently dense family to functions, that is, in terms of partial sections with respect to κ or χ, or by level curves, that is, sections with respect to γ. The values of γ at intermediate points can be approximated with sufficient accuracy by interpolation.

Figures 3(a), 4(a) and 5(a) show level curves of the surface $\gamma = \gamma_{k\alpha}(\varkappa, \chi) = c$, $k = 3, 4, 5$ for the three types of boundary conditions considered; the numbers on the curves indicate the value of c when the parameters κ and χ vary in the ranges $|\varkappa| \leqslant 0.99$, $0 \leqslant \chi \leqslant 0.99$. The case $\kappa = 0$ corresponds to the uniform thread considered in Section 5. Comparison of the families reveal the essential role of the boundary conditions (i.e. the mode of attachment at one or both ends). The boundary conditions of case 3 in Section 5 (the classical case) yield strictly positive values of $\gamma_{3\alpha}(\kappa, \chi)$ (see Fig. 3a), which decrease monotonically as functions of κ and χ as $\varkappa \to 1$, $\chi \to 1$ and increase monotonically as $\varkappa \to -1$, $\chi \to 0$. These conclusions are in agreement with mechanical considerations. In particular, it is interesting to observe that, for any fixed value of $0 \leqslant \chi \leqslant 1$, the frequency of the vibration decreases as κ increases from $\kappa = -1$ to $\kappa = 1$. The graphs in Fig. 4(a) confirm the considerable influence of the boundary conditions: as $\kappa \to 1$ a large part of the thread remains fixed, leading to an increase in the vibration frequency. Note that the decrease as a function of χ in the value of γ for fixed values of κ seems quite

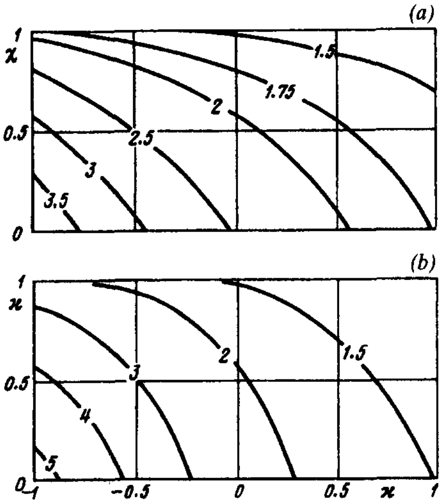

Fig. 3.

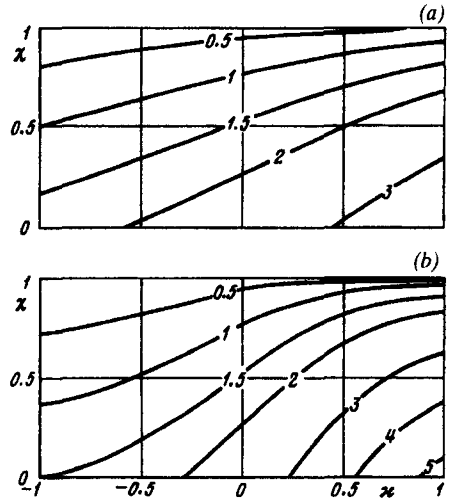

Fig. 4.

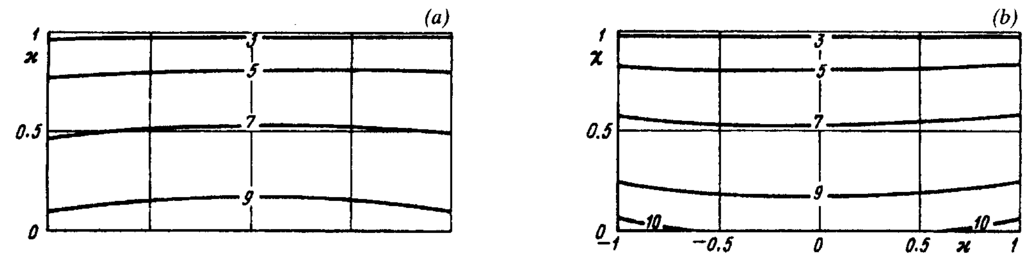

Fig. 5.

obvious. The qualitative difference between this case and that of Fig. 3(a) is that $\gamma_{4a}(\varkappa, 1) \equiv 0$, that is, the vibration "frequency" becomes imaginary ($\lambda_1 = -\omega^2$). This indicates instability of the motions of the thread.

Figure 5(a) represents analogous sections for the boundary conditions of case 3 in Section 5 (both ends are fixed). The curves are almost completely symmetrical about the $\kappa = 0$ axis, confirming the crucial significance of the boundary conditions. One observes only a slight influence of the mass distribution. Note that the values of $\gamma_{5\alpha}(\kappa, \chi)$ for all $|\varkappa| < 1$, $0 \leqslant \chi < 1$ exceed $\gamma_{3,4\alpha}(\kappa, \chi)$; this is a consequence of the influence of the boundary conditions (see below).

6.2. *A parabolically varying linear density.* Let us assume that the cross-section of the thread is a circle whose radius is a linear function of κ, like the function $\rho(x)$ of (6.1). The volume is assumed to remain constant. Then, after introducing a dimensionless argument and dimensionless parameters for the functions $p(x)$ and $r(x)$ in (6.2), we obtain

$$\begin{aligned} p(x) &= 1-\chi[1-(3(1+\varkappa)^2 x - 6(1+\varkappa)\varkappa x^2 + 4\varkappa^2 x^3)(3+\varkappa^2)^{-1}] \\ r(x) &= 3(1-2\varkappa(x-1/2))^2(3+\varkappa^2)^{-1}, \quad 0 \le x \le 1, \quad |\varkappa| < 1, \quad 0 \le \chi < 1 \end{aligned} \tag{6.4}$$

The parameter χ in (6.4) has the same meaning as in (6.2). In the limiting case $\chi = 0$ ($P_0/W_0 \to 0$), the tension is $p(x) \equiv 1$, corresponding to the model of a non-uniform string of variable linear density $r(x)$. The initial approximation to γ in the method of continuation with respect to the parameters κ and χ is defined as in Section 6.1, while the boundary conditions have the form (6.3). Graphs of the computed values of $\gamma_{k6}(\varkappa, \chi)$, $k = 3, 4, 5$ are shown in Figs 3(b), 4(b) and 5(b); they correspond to the three types of boundary conditions indicated in Section 6.1 (see (6.3)). Comparison with the corresponding curves, which were discussed above, shows that their behaviour is qualitatively the same. The numerical data for fixed values of κ, χ ($\chi < 1$) may differ considerably. The somewhat higher values in Figs 3(b) (as $\kappa \to -1$) and 4(b) (as $\kappa \to 1$), and, conversely, the lower values (as $\kappa \to 1$ and as $\kappa \to -1$) reflect the fact that a relatively large or small part of the thread is fixed; this again confirms the essential influence of the boundary conditions.

It is interesting to compare the curves $\gamma_{5b} = c$ and $\gamma_{5a} = c$ (see Figs 5b and 5a). Their qualitative behaviour is the same, but there is a very subtle difference. The slight upward curvature of the curves in Fig. 5(b) as $|\kappa| \to 1$ can be explained by the fact that if the mass distribution has the form (6.4), a relatively smaller part of the thread mass m is engaged in vibration, compared to the case of a uniform thread ($\kappa = 0$). In Fig. 5(a), a downward curvature is observed. This corresponds to the mass distribution (6.2). In this case, a relatively larger part of the thread mass is engaged in vibration.

The stability of the motion of a non-uniform thread can be investigated in the same way as in Section 5 using inequality (5.6). When $\gamma(\varkappa, \chi) > \Gamma_\chi$, the motion is stable; otherwise it is unstable (secular or exponential instability).

The natural vibration and the conditions for its stability and instability, allowing the density of the thread to be arbitrary, may be analysed by analogy with the previous discussion.

Thus, the numerical-analytical method of accelerated convergence proposed here enables one to determine efficiently the natural frequencies and modes of vibration of a heavy non-uniform thread with additional tension for different boundary conditions. If the plane of the vibration is rotating, the method also enables one to determine the range of parameter values, including the velocity of rotation, for which these motions satisfy stability or instability conditions.

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